GROMOV (NON)HYPERBOLICITY OF CERTAIN DOMAINS IN $\mathbb{C}^2$

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ABSTRACT. We prove the non-hyperbolicity of the Kobayashi distance for $\mathcal{C}^{1,1}$-smooth convex domains in $\mathbb{C}^2$ which contain an analytic disc in the boundary or have a point of infinite type with rotation symmetry. Moreover, examples of smooth, non pseudoconvex, Gromov hyperbolic domains are given; we prove that the symmetrized polydisc and the tetrablock are not Gromov hyperbolic and write down some results about Gromov hyperbolicity of product spaces.

1. INTRODUCTION AND STATEMENTS

In [12], Gromov introduced the notion of almost hyperbolic space. He discovered that "negatively curved" space equipped with some distance share many properties with the prototype, even though the distance does not come from a Riemannian metric. This gave the impulse to intensive research to find new interesting classes of spaces which are hyperbolic in that sense. In this paper we are mainly interested in investigating this concept with respect to Kobayashi (pseudo)distance on convex domains (the only exceptions being Propositions 1 and 2). One may suspect that it is a restriction to consider only the Kobayashi metric. Actually, because Carathéodory, Kobayashi and Bergman distances on convex domains, or more generally on $\mathbb{C}$-convex domains containing no complex lines, are bilipschitz equivalent [17, Theorem 12], it does not matter which one we choose (see e.g. [19, Th. 3.18, Th. 3.20]).

Definition 1. *Let (D, d) be a metric space. Given points x, y, $z \in D$,, the* Gromov product *is*

$$(x, y)_z := d(x, z) + d(z, y) - d(x, y).$$

Let

$$S(p, q, x, w) := \min((p, x)_w, (x, q)_w) - (p, q)_w.$$

D is Gromov hyperbolic *with respect to d if and only if*

$$\sup_{p,q,x,w \in D} S(p, q, x, w) < \infty.$$

If $S(p,q,x,w) \leq 2\delta$, we say that D is δ-hyperbolic.

Definition 2. *(D, d) is a* path metric space *if, given any two points x, $y \in D$ and $\epsilon > 0$, there is a rectifiable path joining x and y with length at most $d(x, y) + \epsilon$.*

We refer the interested reader to [5] or [19] for other characterisations of Gromov hyperbolicity. We chose this one because it does not use geodesics explicitly.

From now on, let D be a domain in $\mathbb{C}^n$.

Denote by l_D the Lempert function of D

$$l_D(z, w) = \inf\{\tanh^{-1}|\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) \text{ with } \varphi(0) = z, \varphi(\alpha) = w\},$$

where $\mathbb{D}$ is the unit disc. The Kobayashi distance k_D is the largest pseudodistance not exceeding l_D. Lempert's seminal paper [15] proved that $l_D = k_D$ for convex domains. An important property of k_D is that it is the integrated form of the *Kobayashi metric* κ_D on D, i.e.

$$k_D(z, w)$$
$$= \inf\{\int \kappa_D(\gamma(t), \gamma'(t))dt : \gamma \text{ is a piecewise } \mathcal{C}^1 \text{ curve joining } z, w\},$$

where

$$\kappa_D(z, X) = \inf\{|\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) \text{ with } \varphi(0) = z, \alpha\varphi'(0) = X\},$$

z, $w \in D$, $X \in \mathbb{C}^n$ (we refer to [13] for basic properties of k_D).

The first work concerning Gromov hyperbolicity on domains endowed with Kobayashi distance was given by Balogh and Bonk [4] who gave both positive and negative examples. Among other results, they proved that the Cartesian product of strictly pseudoconvex domains is not Gromov hyperbolic. It is a special case of a general situation mentioned in many places but without proof (cf. [11]).

Proposition 1. *Assume that (X_1, d_1) is a path metric space with d_1 unbounded and (X_2, d_2) a metric space with unbounded d_2. Let $d = \max\{d_1, d_2\}$. Then $(X_1 \times X_2, d)$ is not Gromov hyperbolic.*

The next proposition is more general than the previous one. However its proof uses Proposition 1.

Proposition 2. *Let (X_1, d_1) and (X_2, d_2) be metric spaces, such that one of them is a path metric space. Then $(X_1 \times X_2, d)$ is Gromov hyperbolic if and only if one of the factors is Gromov hyperbolic and the metric of the second one is bounded (in particular, it is also Gromov hyperbolic).*

Moreover, the proof of Proposition 1, and the remark following it, show that the path property in Proposition 2 can be replaced the following.

Definition 3. *A metric space (Y, d) admits the* weak midpoints property *if either d is bounded or there exist sequences $(x_k), (y_k), (z_k) \subset Y$ such that $d(x_k, y_k) \to \infty$ and*

$$(1) \qquad \frac{\max(d(x_k, z_k), d(y_k, z_k))}{d(x_k, y_k)} \to \frac{1}{2}.$$

Corollary 1. *Let D_1 and D_2 be Kobayashi hyperbolic domains admitting non-constant bounded holomorphic functions (for example, bounded domains). Then $D_1 \times D_2$ is not Gromov hyperbolic.*

Note also that if D_1 and D_2 are planar domains with complements containing more than one point, then $D_1 \times D_2$ is not Gromov hyperbolic w.r.t. the Kobayashi distance (use that $k_{D_k}(z, z_j) \to \infty$ as $z_j \to \partial D_k$).

As an immediate consequence we obtain that the polydisc is not hyperbolic. Moreover, even its "symmetrized" counterpart is not.

Proposition 3. $\mathbb{G}_n$ *is not Gromov hyperbolic w.r.t. the Carathéodory and the Kobayashi distances for $n \geq 2$. Moreover, G_2 is not Gromov hyperbolic with respect to Bergman distance.*

For the convenience of the reader, recall that *the symmetrized polydisc* $\mathbb{G}_n$, which is of great relevance due to its properties and role (s.e. [2], [7], [8]), is the image of the holomorphic map (s.e. [18])

$$\pi : \mathbb{D}^n \to \mathbb{C}^n, \ \pi = (\pi_1, \ldots, \pi_n),$$

$$\pi_k(z_1, \ldots, z_n) \sum_{1 \leq j_1 < \ldots < j_k \leq n} z_{j_1} \ldots z_{j_k}, \ z_1, \ldots, z_n \in \mathbb{D}, 1 \leq k \leq n$$

which is proper from $\mathbb{D}^n$ to $\mathbb{G}_n$.

Another interesting domain, the *tetrablock*, fails to be hyperbolic. Let

$$\varphi : \mathcal{R}_{II} \to \mathbb{C}^3, \ \varphi(z_{11}, z_{22}, z) := (z_{11}, z_{22}, z_{11}z_{22} - z^2),$$

where $\mathcal{R}_{II}$ denotes the classical Cartan domain of the second type (in $\mathbb{C}^3$), that is

$$\mathcal{R}_{II} = \{\widetilde{z} \in \mathcal{M}_{2\times 2}(\mathbb{C}) : \widetilde{z} = \widetilde{z}^t, \ \|\widetilde{z}\| < 1\},$$

where $\|\cdot\|$ is the operator norm and $\mathcal{M}_{2\times 2}(\mathbb{C})$ denotes the space of 2×2 complex matrices (we identify a point $(z_{11}, z_{22}, z) \in \mathbb{C}^3$ with a 2×2 symmetric matrix $\begin{pmatrix} z_{11} & z \\ z & z_{22} \end{pmatrix}$). Then φ is a proper holomorphic map and $\varphi(\mathcal{R}_{II}) = \mathbb{E}$ is a domain (s.e. [18, Remark 4]), called the tetrablock.

Proposition 4. $\mathbb{E}$ *is not Gromov hyperbolic.*

Note that $\mathbb{G}_2$ and $\mathbb{E}$ are $\mathbb{C}$-convex.

Buckley in [6], following Bonk, claimed that it is because of the flatness of the boundary rather than the lack of smoothness that Gromov hyperbolicity fails. Recently, Gaussier and Seshadri have provided a proof of that conjecture. More precisely, their main result in [11, Theorem 1.1] states that any bounded convex domain in $\mathbb{C}^n$ whose boundary is $\mathcal{C}^\infty$-smooth and contains an analytic disc, is not Gromov hyperbolic with respect to the Kobayashi distance. Lemma 5.4 in their proof used the $\mathcal{C}^\infty$ assumption in an essential way. Our aim is to prove this result in a shorter way in $\mathbb{C}^2$, assuming only $\mathcal{C}^{1,1}$-smoothness. Moreover, the proofs of the facts we use are more elementary.

Theorem 1. *Let D be a convex domain in $\mathbb{C}^2$ containing no complex lines.[1] Assume that ∂D is $\mathcal{C}^{1,1}$-smooth and contains an analytic disc. Then D is not Gromov hyperbolic with respect to the Kobayashi distance.*

Besides, we give a partial answer to the question raised in [4].

Theorem 2. *Let D be a $\mathcal{C}^{1,1}$-smooth convex bounded domain in $\mathbb{C}^2$ admitting a defining function of the form $\varrho(z) = -\Re z_1 + \psi(|z_2|)$ near the origin, where ψ is a $\mathcal{C}^{1,1}$-smooth nonnegative convex function near 0 satisfying: $\psi(0) = 0$, and*

(2)
$$\limsup_{x \to 0} \frac{\log \psi(|x|)}{\log |x|} = +\infty.[2]$$

Then, D is not Gromov hyperbolic.

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[1]Then D is biholomorphic to a bounded domain (s.e. [13, Theorem 7.1.8]).

[2]If ψ is $\mathcal{C}^\infty$, the condition (2) means that 0 is of infinite type.

There naturally arises the question whether there is any connection between Gromov hyperbolicity and pseudoconvexity. The known examples do not say anything in this matter. Also, it is easy to construct domains which are Gromov hyperbolic but neither pseudoconvex nor smooth. Indeed, take any strictly pseudoconvex domain G. Assume that A is a relatively closed subset of G with $H^{2n-2}(A) = 0$, where H^{2n-2} denotes the $(2n-2)$-dimensional Hausdorff measure. To prove our claim, it remains to notice that

$$k_G|_{(G\setminus A)\times(G\setminus A)} = k_{G\setminus A}$$

(cf. [13, Theorem 3.4.2]) and to apply [4, Theorem 1.4].

The above example does not have a smooth boundary. The next proposition yields, in particular, a family of non pseudoconvex domains with smooth boundaries which are Gromov hyperbolic.

Proposition 5. *Let $G \subset \mathbb{C}^n$, $n \geq 2$ be a bounded strictly pseudoconvex domain and $D \Subset G$ have one of the following form:*

- *D is $\mathcal{C}^2$-smooth and its Levi form has at least one strictly positive eigenvalue at each boundary point.*
- *D is a polydisc.*

Then $G \setminus \overline{D}$ is Gromov hyperbolic.

Observe that the Riemann Singularity Removable Theorem with [4, Theorem 1.4] offers another possibility to achieve a similar statement to the one above for the Bergman or the Carathéodory distances. Observe also that in the second case the domain has some flat part in its boundary.

Throughout the paper d_D denotes the (Euclidean) distance to ∂D. A point $z \in \mathbb{C}^n$ we write as $(z_1, \ldots, z_n)$, $z_j \in \mathbb{C}$.

2. PROOFS

Proof of Proposition 1.

Assume that d is $\frac{\delta}{2}$-Gromov hyperbolic. Put $k = 3 + \delta$. Then there are points $y_1, y_2 \in X_2$ such that $d_2(y_1, y_2) = 2s \geq 2k$. Choose points $x_1, x_2^* \in X_1$ with $d_1(x_1, x_2^*) \geq 2s$. By the path property of X_1, there is a d_1-continuous curve $\gamma : [0, 1] \to X_1$ joining the points x_1 and x_2^* such that $L_{d_1}(\gamma) < d_1(x_1, x_2^*) + 1$. Note that $t \to d_1(x_1, \gamma(t))$ is continuous. Hence there is a smallest t_0 such that $d_1(x_1, \gamma(t_0)) = 2s$. Set $x_2 := \gamma(t_0)$.

Now $L(\gamma|_{[0,t_0]}) \geq d_1(x_1, x_2) = 2s$, and

$$L(\gamma|_{[0,t_0]}) = L(\gamma) - L(\gamma|_{[t_0,1]}) \leq d_1(x_1, x_2^*) + 1 - d_1(x_2, x_2^*) \leq d_1(x_1, x_2) + 1.$$

Let t_1 be the smallest number in $[0, t_0]$ such that $d_1(x_1, \gamma(t_1)) = s$. Set $x_3 := \gamma(t_1)$. Then

$$d_1(x_2, x_3) \geq d_1(x_1, x_2) - d_1(x_1, x_3) = s, \text{ and}$$

$$d_1(x_2, x_3) = L(\gamma|_{[0,t_1]}) = L(\gamma|_{[0,t_0]}) - L(\gamma|_{[t_1,t_0]}) \leq 2s + 1 - d_1(x_1, x_2) = s + 1.$$

Hence, $s = d_1(x_1, x_3) \leq d_1(x_3, x_2) < s + 1$.

Now define the following points in $X_1 \times X_2$: $x := (x_1, y_1)$, $y := (x_2, y_1)$, $w := (x_3, y_1)$, and $z := (x_3, y_2)$. Then $d(z, w) = d(z, x) = d(z, y) = 2s$ and $(x, y)_w \leq 1$, $(x, z)_w = d(x, w) = s$, $(y, z)_w = d(y, w) \geq s - 1$. By the assumption of $\frac{\delta}{2}$-hyperbolicity we reach the following inequality

$$1 \geq (x, y)_w \geq \min\{(y, z)_w, (x, z)_w\} - \delta \geq s - 1 - \delta \geq 2;$$

a contradiction.

Remark 1. *An essential ingredient in the proof of Proposition 1 is the existence of points x_1, x_2, x_3 such the triangle inequality is a near-equality, namely $(x_1, x_2)_{x_3} \leq 1$. The condition (1) is equivalent to $(x_1, x_2)_{x_3} = o(d(x_1, x_2))$, and $|d(x_1, x_3) - d(x_2, x_3)| = o(d(x_1, x_2))$.*

Using this weaker hypothesis and following the steps of the above proof, setting $2s = d(x_1, x_2)$ as before, we find

$$o(s) \geq (x, y)_w \geq s - o(s) - \delta,$$

leading to a contradiction when $s \to \infty$. Similar changes can be made in the proof below.

Proof of Proposition 2.

Let first X_1 be 2δ-hyperbolic and $d_2 \leq 2c$. Since $d \leq d_1 + 2c$, it follows that

$$(x_1, y_1)_{w_1} - 2c \leq (x, y)_w \leq (x_1, y_1)_{w_1} + 4c$$

and then (X, d) is $(\delta + 3c)$-hyperbolic.

Assume now that (X, d) is δ-hyperbolic. Following the proof of Proposition 1, we deduce that one of the distances is bounded, say $d_2 \leq 2c$. Then we get as above that (X_1, d_1) is $(\delta + 3c)$-hyperbolic.

Proof of Corollary 1.

It is enough to observe that if G admits a non-constant bounded holomorphic function f and $|f(z_j)| \to \sup_G |f|$, then $k_G(z, z_j) \geq c_G(z, z_j) \to \infty$.

Proof of Proposition 3.

Fix $a \in \mathbb{D}$. Put $p_a = \pi(a, \dots, a)$, $q_a = \pi(a, \dots, a, -a)$, $m_a = \pi(a, \dots, a, 0)$. Holomorphic contractibility and product property gives the following

$$k_{\mathbb{D}}(\pi_n(z), \pi_n(w)) \leq k_{\mathbb{G}_n}(\pi(z), \pi(w))$$
$$\leq \inf\{k_{\mathbb{D}^n}(\widetilde{z}, \widetilde{w}) : \pi(z) = \pi(\widetilde{z}), \ \pi(w) = \pi(\widetilde{w})\}$$
$$= \inf\left\{ \inf_{1 \leq j \leq n} k_{\mathbb{D}}(\widetilde{z}_j, \widetilde{w}_j), \pi(z) = \pi(\widetilde{z}), \ \pi(w) = \pi(\widetilde{w})\right\}, \ z, w \in \mathbb{D}^n.$$

Consequently,

$$\liminf_{a \to \partial \mathbb{D}}[k_{\mathbb{G}_n}(p_a, q_a) - k_{\mathbb{G}_n}(q_a, 0) - k_{\mathbb{G}_n}(p_a, m_a)],$$
$$\liminf_{a \to \partial \mathbb{D}}[k_{\mathbb{G}_n}(p_a, q_a) - k_{\mathbb{G}_n}(p_a, 0) - k_{\mathbb{G}_n}(q_a, m_a)] > -\infty,$$

and finally

$$(p_a, m_a)_0 - (p_a, q_a)_0, \ (q_a, m_a)_0 - (p_a, q_a)_0 = k_{\mathbb{G}_n}(m_a, 0) + \mathrm{O}(1).$$

It remains to recall the fact that $\mathbb{G}_n$ is a c-finite compact domain (see [8, Corollary 3.2]).

The last part follows from $\mathbb{C}$-convexity of $\mathbb{G}_2$.

Proof of Proposition 4.

Let $a \in (0, 1)$, and put $P_a = \mathrm{diag}(a, a)$, $Q_a = \mathrm{diag}(a, -a)$. Recall that $\Phi_a(Z) = (Z - a\mathrm{I})(\mathrm{I} - aZ)^{-1}$ is an automorphism of $\mathcal{R}_{II}$. Direct computation shows that

$$\varphi \circ \Phi_a\left(\begin{pmatrix} z_{11} & z \\ z & z_{22} \end{pmatrix}\right) = \varphi \circ \Phi_a\left(\begin{pmatrix} z_{11} & -z \\ -z & z_{22} \end{pmatrix}\right),$$

whenever $\begin{pmatrix} z_{11} & z \\ z & z_{22} \end{pmatrix} \in \mathcal{R}_{II}$. Thus, Φ_a induces an automorphism $\widetilde{\Phi}_a$ of $\mathbb{E}$. Because of this and [1]

$$k_{\mathbb{E}}(0, (a, b, p)) = \tanh^{-1} \max\left\{\frac{|a - \bar{b}p| + |ab - p|}{1 - |b|^2}, \frac{|b - \bar{a}p| + |ab - p|}{1 - |a|^2}\right\},$$
$$(a, b, p) \in \mathbb{E},$$

$$k_{\mathbb{E}}(P_a, 0), \ k_{\mathbb{E}}(Q_a, 0), \ \frac{1}{2}k_{\mathbb{E}}(P_a, Q_a) = -\frac{1}{2}\log d_{\mathbb{D}}(a) + \mathrm{O}(1).$$

Observe that $g_a = \widetilde{\Phi}_{-a} \circ f$, where $f(\lambda) = (0, \lambda, 0)$, is a geodesic s.t. $P_a, Q_a \in \mathrm{im}\, f$. The (Kobayashi) middle point of $g_a|_{[\frac{-2a}{1+a^2}, 0]}$ tends to the boundary. Precisely

$$g_a(0, -a, 0) \to \mathrm{diag}(1, 0) \text{ if } a \to 1$$

and it finishes the proof (after application of [16, Proposition 2]).

Proof of Theorem 1.

Since ∂D contains an analytic disc, it is well known that it contains an affine disc (cf. [17, Proposition 7]). We assume that this disc has center 0 and lies in $\{z_1 = 0\}$, and that $D \subset \{\Re z_1 > 0\}$.

Lemma 1. *We can find an $r > 0$ such that for any $\delta > 0$ small enough there exist two discs $\Delta(\tilde{p}_\delta, r)$ and $\Delta(\tilde{q}_\delta, r)$ in $D_\delta := D \cap \{z_1 = \delta\}$ which touch ∂D at two points $\hat{p}_\delta$ and $\hat{q}_\delta$ with $\|\hat{p}_\delta - \hat{q}_\delta\| > 5r$.*

Proof. We identify $\partial D \cap \{z_1 = 0\}$ with a closed, bounded, convex subset of $\mathbb{C}$, which is the closure of its interior. Call this interior D_0.

There exists $\zeta_0 \in D_0$ s.t. $\operatorname{dist}(\zeta_0, \mathbb{C} \setminus D_0) = \max_{\zeta \in D_0} \operatorname{dist}(\zeta, \mathbb{C} \setminus D_0)$. Then the set

$$M := \left\{ p \in \partial D_0 : |p - \zeta_0| = \min_{\xi \in \partial D_0} |\xi - \zeta_0| \right\}$$

is not empty and cannot be contained in any half plane $H_\theta := \{\zeta : \Re[(\zeta - \zeta_0)e^{-i\theta}] < 0\}$. If it were, one could find $\varepsilon > 0$ s.t. $\operatorname{dist}(\zeta_0 + \varepsilon e^{i\theta}, \mathbb{C} \setminus D_0) > \operatorname{dist}(\zeta_0, \mathbb{C} \setminus D_0)$. So there are $\hat{p} \neq \hat{q} \in M$ such that $\arg((\hat{p} - \zeta_0)(\hat{q} - \zeta_0)^{-1}) \geq 2\pi/3$. Take $r \in (0, \frac{\sqrt{3}}{5+\sqrt{3}}|\hat{p} - \zeta_0|)$, $\tilde{p} := \zeta_0 + (1 - r|\hat{p} - \zeta_0|^{-1})(\hat{p} - \zeta_0)$, and $\tilde{q}$ chosen likewise. Then $\Delta(\tilde{p}, r), \Delta(\tilde{q}, r) \subset D_0$ and are tangent to ∂D_0 at $\hat{p}$ and $\hat{q}$.

Now we want to move the discs we have constructed inside the domain. By $\mathcal{C}^{1,1}$-smoothness of D, we can move them (in $\mathbb{C}^2$) along the vector $(1,0)$ inside D, that is $\Delta(\tilde{p}, r), \Delta(\tilde{q}, r) \subset D \cap \{z_1 = \delta\} = D_\delta$, for $0 < \delta < \delta_0$. If they do not touch ∂D_δ, then shift them (separately at every sublevel set) to the boundary but leaving their centers on the real line passing through $\tilde{p} + (\delta, 0)$ and $\tilde{q} + (\delta, 0)$. Denote new discs by $\Delta(\tilde{p}_\delta, r), \Delta(\tilde{q}_\delta, r)$, and by $\hat{p}_\delta, \hat{q}_\delta$ points of contact of those discs with ∂D_δ. $\qquad\square$

Set $\tilde{s}_\delta = \frac{\tilde{p}_\delta + \tilde{q}_\delta}{2}$.

Choose now a point $a = (\delta_0, 0) \in D$ ($\delta_0 > 0$) and consider the cone with vertex at a and base $\partial D \cap \{z_1 = 0\}$. Denote by G_δ the intersection of this cone and $\{z_1 = \delta\}$. For any δ small enough the line segment with ends at $\tilde{p}_\delta$ and $\hat{p}_\delta$ intersects ∂G_δ, say at p_δ. Define q_δ in a similar way.

We shall show that $S(p_\delta, q_\delta, \tilde{s}_\delta, a) \to +\infty$ as $\delta \to 0$. For this we will see that $(p_\delta, \tilde{s}_\delta)_a - (p_\delta, q_\delta)_a \to +\infty$ as $\delta \to 0$. It will follow in the same way that $(q_\delta, \tilde{s}_\delta)_a - (p_\delta, q_\delta)_a \to +\infty$.

It is enough to prove that

(3) $$k_D(q_\delta, a) - k_D(\tilde{s}_\delta, a) < c_1$$

and

(4) $$k_D(p_\delta, q_\delta) - k_D(p_\delta, \tilde{s}_\delta) \to +\infty.$$

Here and below $c_1, c_2, \ldots$ denote some positive constants which are independent of δ.

For (3), observe that convexity and smoothness implies that

$$(5) \qquad k_D(\tilde{s}_\delta, a) \geq \frac{1}{2} \log \frac{d_D(a)}{d_D(\tilde{s}_\delta)} \text{ and } 2k_D(q_\delta, a) \leq -\log d_D(q_\delta) + c_2$$

(cf. [16, (2)] and [13, Proposition 10.2.3][3]). It remains to use that $d_D(\tilde{s}_\delta) = d_D(q_\delta)$ for any $\delta > 0$ small enough.

To prove (4), denote by F_δ the convex hull of $\Delta(\tilde{p}_\delta, r)$ and $\Delta(\tilde{s}_\delta, r)$. Then by inclusion $k_D(p_\delta, \tilde{s}_\delta) \leq k_{F_\delta}(p_\delta, \tilde{s}_\delta)$.

Claim. $k_{F_\delta}(p_\delta, \tilde{s}_\delta) < -\frac{1}{2} \log d'_D(p_\delta) + c_3$, where d'_D is the distance to ∂D in the z_2-direction.

Indeed, for δ small enough

$$d'_D(p_\delta) = d_{D_\delta}(p_\delta) = d_{F_\delta}(p_\delta) = d_{\Delta(\tilde{p}_\delta, r)}(p_\delta)$$

because the closest point on ∂D_δ belongs to $\partial \Delta(\tilde{p}_\delta, r)$. Now $k_{F_\delta}(p_\delta, \tilde{s}_\delta) \leq k_{F_\delta}(p_\delta, \tilde{p}_\delta) + k_{F_\delta}(\tilde{p}_\delta, \tilde{s}_\delta)$.

By explicit computations in the circle,

$$k_{F_\delta}(p_\delta, \tilde{p}_\delta) \leq -\frac{1}{2} \log d_{\Delta(\tilde{p}_\delta, r)}(p_\delta) + C(r) = -\frac{1}{2} \log d'_D(p_\delta) + C(r).$$

On the other hand, by using a finite chain of disks of radius r with centers on the line segment from $\tilde{p}_\delta$ to $\tilde{s}_\delta$, $k_{F_\delta}(\tilde{p}_\delta, \tilde{s}_\delta) \leq C \frac{|\tilde{p}_\delta - \tilde{q}_\delta|}{r} \leq C(r)$. The desired assertion follows.

We shall show that

$$(6) \qquad 2k_D(p_\delta, q_\delta) > -\log d'_D(p_\delta) - \log d'_D(q_\delta) - c_4,$$

which implies (4), because $d'_D(q_\delta) \to 0$ as $\delta \to 0$.

Since the Kobayashi distance is the integrated form of the Kobayashi metric, we may find a point $m_\delta \in D$ s.t.

$$\|p_\delta - m_\delta\| = \|q_\delta - m_\delta\| \geq \frac{\|p_\delta - q_\delta\|}{2},$$

$$k_D(p_\delta, q_\delta) > k_D(p_\delta, m_\delta) + k_D(m_\delta, q_\delta) - 1.$$

Let $\check{p}_\delta \in \partial D$ be the closest point to p_δ in the direction of the complex line through p_δ and m_δ.

By [17, (4)], there exists a constant $C > 0$ s.t. for every convex domain D in $\mathbb{C}^2$, for any unit vector X

$$\frac{1}{d_X(z)} \leq \frac{|\langle e_1(z), X \rangle|}{d_1(z)} + \frac{|\langle e_2(z), X \rangle|}{d_2(z)} \leq C \frac{1}{d_X(z)},$$

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[3]$\mathcal{C}^2$-smoothness is assumed there but only the locally uniform interior ball condition is used.

where $\{e_1(z), e_2(z)\}$ is a minimal basis at z, and $\{d_j(z)\}$, $j = 1, 2$, $d_X(z)$ are the respective distances in directions $\{e_j(z)\}$, $j = 1, 2$, X. Since $d_1 \leq d_2$, this implies

$$(7) \qquad 1 \leq |\langle e_1(z), X \rangle| + |\langle e_2(z), X \rangle| \leq C \frac{d_2(z)}{d_X(z)}.$$

Let $X := \frac{m_\delta - p_\delta}{\|m_\delta - p_\delta\|}$, then $\|p_\delta - \check{p}_\delta\| = d_X(p_\delta)$, $d'_D(p_\delta) = d_2(p_\delta)$, so (7) translates to

$$(8) \qquad \|p_\delta - \check{p}_\delta\| < c_5 d'_D(p_\delta).$$

By convexity, D is on the one of the sides, say H_δ, of the real tangent plane to ∂D at $\check{p}_\delta$. Hence, since $\frac{\|m_\delta - \check{p}_\delta\|}{d_{H_\delta}(m_\delta)} = \frac{\|p_\delta - \check{p}_\delta\|}{d_{H_\delta}(p_\delta)}$, by (5),

$$(9) \qquad 2k_D(p_\delta, m_\delta) \geq 2k_{H_\delta}(p_\delta, m_\delta) \geq \log \frac{d_{H_\delta}(m_\delta)}{d_{H_\delta}(p_\delta)} = \log \frac{\|m_\delta - \check{p}_\delta\|}{\|p_\delta - \check{p}_\delta\|},$$

And by the triangle inequality, and (8),

$$\log \frac{\|m_\delta - \check{p}_\delta\|}{\|p_\delta - \check{p}_\delta\|} \geq$$

$$\log \frac{\|m_\delta - p_\delta\| - \|p_\delta - \check{p}_\delta\|}{\|p_\delta - \check{p}_\delta\|} \geq \log\left(\frac{r}{2\|p_\delta - \check{p}_\delta\|} - 1\right) \geq \log \frac{r}{2c_5 d'(p_\delta)} - 1,$$

for any $\delta > 0$ small enough. Recall now that $d_D(p_\delta)$ is attained in the z_1-direction for any $0 < \delta \ll 1$. So $2k_D(p_\delta, m_\delta) > -\log d'_D(p_\delta) - c_6$. Similarly $2k_D(q_\delta, m_\delta) > -\log d'_D(p_\delta) - c_6$, which implies (6), and completes the proof.

Remark 2. *All the above arguments hold in $\mathbb{C}^n$, $n \geq 3$, except (8).*

Proof of Theorem 2.

Since the case when $\psi(z_0) = 0$ for some $z_0 \neq 0$, is covered by Proposition 1, we may assume that $\psi^{-1}(0) = \{0\}$. Also assume $p = (1, 0) \in D$.

Let $\alpha(x)$, small enough, an increasing function s.t. for any $x > 0$, $\psi'(x) \geq \psi'((1 - \alpha(x))x) \geq \frac{1}{2}\psi'(x)$. We choose, for $x > 0$, $q(x) = (\psi(x), 0)$, $r(x) = (\psi(x), -(1 - \alpha(x))x)$, $s(x) = (\psi(x), (1 - \alpha(x))x)$.

We claim that for x small enough:

(I) $d_D(q) = \psi(x)$,
(II) $\frac{\alpha(x)}{4} x \psi'(x) \leq d_D(s), d_D(r) \leq \alpha(x) x \psi'(x)$,
(III) the functions $k_D(s, q) + \frac{1}{2} \log \alpha(x)$ and $k_D(r, q) + \frac{1}{2} \log \alpha(x)$ are bounded,
(IV) the function $k_D(r, s) + \log \alpha(x)$ is bounded.

Before we proceed to prove the claims we make some general observation about infinite order of vanishing.

Lemma 2. *For any $\varepsilon > 0$ and $A > 0$, there exists $x \in (0, \varepsilon)$ such that $\frac{x\psi'(x)}{\psi(x)} > A$.*

Proof. Suppose instead that there exist $\varepsilon > 0$ and $A > 0$ such that $\frac{x\psi'(x)}{\psi(x)} \leq A$ for $0 < x \leq \varepsilon$. Then

$$\frac{d}{dx}\left(\log \psi(x)\right) \leq \frac{A}{x}, \quad 0 < x \leq \varepsilon,$$

so $\log(\psi(\varepsilon)) - \log(\psi(x)) \leq A\left(\log \varepsilon - \log x\right)$, i.e.

$$\psi(x) \geq \frac{\psi(\varepsilon)}{\varepsilon^A} x^A, \quad 0 < x \leq \varepsilon,$$

which means that at the point 0 there is finite order of contact with the tangent hyperplane, a contradiction. $\qquad\square$

Assume for a while the claims and observe that for any x verifying the conclusion of the observation we have

$$(r, p)_q - (r, s)_q, \ (p, s)_q - (r, s)_q \geq -\frac{1}{2}\log\frac{\psi(x)}{x\psi'(x)} + C_1.$$

Since the above quantity can be made arbitrarily large, it finishes the proof.

It remains to prove (I)-(IV).

(I) is clear. Next, since $(\psi((1 - \alpha(x))x), (1 - \alpha(x))x) \in D$, $d_D(s) \leq \psi(x) - \psi((1 - \alpha(x))x)) \leq \alpha(x)x\psi'(x)$ by convexity. Let L be the real line through $(\psi((1 - \alpha(x))x), (1 - \alpha(x))x)$ and $(\psi(x), x)$. Its slope is less than $\psi'(x)$, so $d_D(s) \leq \mathrm{dist}\,(s, L')$, where L' is the line through $(\psi((1 - \alpha(x))x), (1 - \alpha(x))x)$ with slope $\psi'(x)$, so

$$d_D(s) \geq \frac{\psi(x) - \psi((1 - \alpha(x))x)}{\sqrt{1 + \psi'(x)^2}}$$
$$\geq \frac{1}{2}\alpha(x) \times \psi'((1 - \alpha(x))x) \geq \frac{1}{4}\alpha(x) \times \psi'(x).$$

Thus, $\frac{\alpha(x)}{4}x\psi'(x) \leq d_D(s) \leq \alpha(x)x\psi'(x)$. Analogous estimates hold for r, which gives (II).

The analytic disc $\zeta \mapsto (\psi(x), x\zeta)$ provides immediate upper estimates in (III) and (IV).

To get lower estimate for $k_D(s, q)$, we map D to a domain in $\mathbb{C}$ by the complex affine projection π_s to $\{z_1 = \psi(x)\}$, parallel to the complex tangent space to ∂D at the point $(\psi(x), x)$. Then $\pi_s(D) = \{\psi(x)\} \times D_s$, where D_s is a convex domain in $\mathbb{C}$, containing the disk $\{|z_2| < x\}$, and

its tangent line at the point x is the real line $\{\Re z_2 = x\}$. The projection is given by the explicit formula

$$\pi_s(z_1, z_2) = \left(\psi(x), z_2 + \frac{\psi(x) - z_1}{\psi'(x)}\right).$$

We renormalize by setting $f_+(z) = 1 - \frac{1}{x}[\pi_s(z)]_2$. Therefore
(10)
$$k_D(s, q) \geq k_{D_s}((1 - \alpha(x))x, 0) \geq k_H(f_+(s), f_+(q)) \geq -\frac{1}{2}\log\alpha(x) + C_2,$$

where $C_2 > 0$ does not depend on x, $H = \{z \in \mathbb{C} : \Re z > 0\}$.

The estimate for $k_D(r, q)$ proceeds along the same lines, but we use the projection π_r to $\{z_1 = \psi(x)\}$ along the complex tangent space to ∂D at $(\psi(x), x)$, given by

$$\pi_r = \left(\psi(x), z_2 - \frac{\psi(x) - z_1}{\psi'(x)}\right).$$

Note that choosing $f_-(z) = 1 + \frac{1}{x}[\pi_r(z)]_2$, we have $f_-(D) \subset \{\Re z > 0\}$.

Now we tackle the lower estimates for $k_D(r, s)$. Let γ be any piecewise $\mathcal{C}^1$ curve s.t. $\gamma(0) = r$, $\gamma(1) = s$. Let $c_0 < \frac{1}{2}$. We claim that there exists $t_0 \in (0, 1)$ s.t. if we set $u = \gamma(t_0)$, then $|f_+(u)|, |f_-(u)| \geq c_0$.

For this write $\gamma = (\gamma_1, \gamma_2)$. Set $\zeta_1 = 1 - \frac{\psi(x)}{x\psi'(x)}$. By the explicit form of π_s, the condition $|f_+(u)| \geq c_0$ reads $|\zeta_1 - \frac{\gamma_2}{x}| \geq c_0$, and the condition $|f_-(u)| \geq c_0$ reads $|\zeta_1 + \frac{\gamma_2}{x}| \geq c_0$. We claim that the disks $\overline{\mathbb{D}}(\zeta_1, c_0)$ and $\overline{\mathbb{D}}(-\zeta_1, c_0)$ are disjoint for any t. Indeed, they would intersect if and only if $0 \in \overline{\mathbb{D}}(\zeta_1, c_0)$, which implies

$$\Re\left(\frac{\gamma_1}{x}\right) \leq -1 + c_0 + \frac{\psi(x)}{x\psi'(x)} \leq -\frac{1}{3}$$

for any x s.t. $\frac{\psi(x)}{x\psi'(x)} \leq \frac{1}{6}$, which we may assume by Lemma 2. In particular $\Re\gamma_1 < 0$, which is excluded for any $\gamma(t) \in D$. Now let $t_1 = \max\{t : \frac{\gamma_2(t)}{x}\} \in \overline{\mathbb{D}}(\zeta_1, c_0)\}$. Then $\frac{\gamma_1(t_1)}{x} \notin \overline{\mathbb{D}}(-\zeta_1, c_0)\}$, and by continuity there is $\eta > 0$ s.t. $\frac{\gamma_1(t_1+\eta)}{x} \notin \overline{\mathbb{D}}(-\zeta_1, c_0)$, and of course $\frac{\gamma_1(t_1+\eta)}{x} \notin \overline{\mathbb{D}}(\zeta_1, c_0)$ by maximality of t_1, so $t_0 = t_1 + \eta$ will do.

Consequently, taking a curve γ such that $k_D(r, s) + 1 \geq \int_0^1 \kappa_D(\gamma(t), \gamma'(t))dt$,

$$\int_0^1 \kappa_D(\gamma(t), \gamma'(t))dt \geq \int_0^{t_0} \kappa_D(\gamma(t), \gamma'(t))dt + \int_{t_0}^1 \kappa_D(\gamma(t), \gamma'(t))dt$$
$$\geq k_D(r, u) + k_D(u, s),$$

and proceeding as in (10) we end the proof.

Proof of Proposition 5.

First, assume that the Levi form of D has at every boundary point at least one positive eigenvalue. Since by [4], every stricty pseudoconvex set is Gromov hyperbolic with respect to the Kobayashi distanace, it remains to be show the boundedness of the function $k_{G\setminus\overline{D}} - k_G$ on $(G\setminus\overline{D}) \times (G\setminus\overline{D})$.

Near the inner boundary, it follows from the estimates for the Kobayshi metric in [14] and [9] for $\mathbb{C}^2$ and $\mathbb{C}^n$, respectively. Recall

$$(11) \qquad \kappa_{G\setminus\overline{D}}(z,\nu_z) \approx d_{G\setminus\overline{D}}(z)^{-\frac{3}{4}},$$

ν_z is a normal vector at z.

Let $D^\varepsilon := \{z \in G\setminus\overline{D} : \text{dist}(z,\overline{D}) \le \varepsilon\}$. For $\varepsilon > 0$ small enough, $D^\varepsilon \Subset G$, and $D^\varepsilon = \{\zeta + t\nu_\zeta : 0 < t \le \varepsilon\}$, where ν_ζ is the outside unit normal vector to ∂D at ζ.

Let $K^\varepsilon := \{z \in G\setminus\overline{D} : \text{dist}(z,\overline{D}) = \varepsilon\}$. It is a compact subset of $G\setminus\overline{D}$ on which $k_{G\setminus\overline{D}} - k_G$ is clearly bounded. By (11), all points in D^ε stand a bounded distance away from K^ε in the $k_{G\setminus\overline{D}}$ distance, and obviously in the k_G distance, too. So, the difference between those two distances cannot become unbounded near the inner boundary.

Now assume to get a contradiction that there exist sequences $\{z_\mu\}$, $\{w_\mu\} \subset G\setminus\overline{D}$ s.t. $k_{G\setminus\overline{D}}(z_\mu, w_\mu) - k_G(z_\mu, w_\mu) \to +\infty$ as $\mu \to \infty$. Without loss of generality, we may assume that $z_\mu \to z$, $w_\mu \to w$, $z, w \in \overline{G}\setminus D^\varepsilon$. When $z \ne w$, [10, Proposition 2.5], [10, Theorem 2.3] gives, respectively, estimate from above if $z, w \in \partial G$, and if either $z \notin \partial G$ or $w \notin \partial G$. On the other hand, the application of [10, Corollary 2.4] provides the behaviour of $k_{G\setminus\overline{D}}$ near (z,w). To complete the proof it this case, note that the estimates derived for $k_{G\setminus\overline{D}}$ are the same as for k_G.

We are left with the case $z = w \in \overline{G}\setminus G$. The proof of [4, Proposition 1.2] says that [4, (1.3)] has local character. Thus [4, Theorem 1.1] remains true for $G\setminus\overline{D}$ near the boundary of G. But, simultaneously the conclusion of [4, Theorem 1.1] holds for G, and it solves the remaining situation.

For the case where D is a polydisc, since all the distances considered are holomorphically invariant, we might assume that $D = \mathbb{D}^n$. All the above arguments work except now we do not know the behaviour of Kobayashi metric near the inner boundary. However, we might proceed as follows. Let $r > 0$ s.t. $(1+r)\overline{\mathbb{D}}^n \subset G$. Take some $z^0 \in [(1+\varepsilon)\mathbb{D}^n]\setminus\overline{\mathbb{D}}^n$, where $0 < 3\varepsilon < r$. Thus, $1+\varepsilon \ge |z_j^0| > 1$ for some $1 \le j \le n$. Observe that

$$\inf_{z\in\partial G,\, w\in\partial((1+\varepsilon)\mathbb{D}^n)} |z - w| > 2\varepsilon.$$

Choose any point $w(z^0) \in \{z_j = z_j^0\} \cap \partial((1+2\varepsilon)\mathbb{D}^n)$, which realizes the distance z^0 to $\partial((1+2\varepsilon)\mathbb{D}^n)$. Let $\Omega_\varepsilon := \{z_j = z_j^0, |z_k| < 1 + 3\varepsilon \text{ for } k \neq j\} \subset \mathbb{C}^{n-1}$. With our choices, we have

$$k_{G \setminus \overline{D}}(z^0, w(z^0)) \leq$$

$$\inf \left\{ k_{\Omega_\varepsilon}(z, w) : z_j = w_j = z_j^0; |z_k| \leq 1 + \varepsilon, |w_k| \geq 1 + 2\varepsilon, k \neq j \right\} < M,$$

for some finite number M which does not depend on z^0. The connectness of $\partial((1+2\varepsilon)\mathbb{D}^n)$ ends the proof.

References

[1] Abouhajar, A., White, M., Young, N.: A Schwarz lemma for a domain related to μ-synthesis, Journal of Geometric Analysis **17** (2007), no. 4, 717–750.

[2] Agler, J., Young, N.J.:The two-by-two spectral Nevanlinna-Pick problem, Trans. Amer. Math. Soc., **356**, 573–585 (2003).

[3] Agler, J., Young, N.J.: A Schwarz lemma for the symmetrized bidisc. Bull. London Math. Soc., **33**, 175-186 (2001).

[4] Balogh, Z.M., Bonk, M.: Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains, Comment. Math. Helv. **75** (2000), 504–533.

[5] Bowditch, B.H.: Notes on Gromov's hyperbolicity criterion for path-metric spaces in: Group theory from a geometrical viewpoint, ed. E. Ghys, A. Haefliger, A. Verjovsky, World Scientific, Singapore, 64–167 (1991).

[6] Buckley, S.: Gromov hyperbolicity of invariant metrics in Research Problems Section Complex variables, Function Spaces and Operators between them.

[7] Costara, C.: On the 2×2 spectral Nevanlinna-Pick problem. J. London Math. Soc.,**71** (2005), 684–702.

[8] Costara, C.: On the spectral Nevanlinna-Pick problem, Studia Math. **170** (2005), 23–55.

[9] Fornaess, J.E., Lee, L.: Kobayashi, Carathéodory and Sibony metric, **54** (2009), 293-301.

[10] Forstneric, F., Rosay, J-P.: Localization of the Kobayashi metric and the boundary continuity of proper holomorphic mappings. Math. Ann. **279** (1987), 239–252.

[11] Gaussier, H., Seshadri, H., On the Gromov hyperbolicity of convex domains in $\mathbb{C}^n$, arXiv:1312.0368.

[12] Gromov, M.: Hyperbolic groups. Essays in group theory, 75–263, Math. Sci. Res. Ins. Publ., 8, Springer, New York, 1987.

[13] Jarnicki, M., Pflug, P.: Invariant distances and metrics in complex analysis, de Gruyter Exp. Math. 9, de Gruyter, Berlin, 1993.

[14] Krantz, S.G.: The boundary behavior of the Kobayashi metric, Rocky. Mount. J. Math., **22** (1992), 227–233.

[15] Lempert, L.: La métrique de Kobayashi et la représentation des domaines sur la boule, Bull. Soc. Math. France **109** (1981), no. 4, 427–474.

[16] Nikolov, N.: Estimates of invariant distances on convex domains, DOI 10.1007/s10231-013-0345-7.

[17] Nikolov, N., Pflug, P., Zwonek, W.: Estimates for invariant metrics on $\mathbb{C}$-convex domains, Trans. Amer. Math. Soc., **363** (2011), 6245–6256.

[18] Trybuła, M.: Proper holomorphic mappings, Bell's formula and the Lu Qi-Keng problem on the tetrablock, Arch. der Math., **101**(6) (2013), 549–558.

[19] Väisälä, J.: Gromov hyperbolic spaces, Expo. Math., **23** (2005), 187–231.

[20] Zwonek, W.: Geometric properties of the tetrablock, Arch. der Math., **100**(2) (2013), 159–165.

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